Graph structure and soficity

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DDC Seminar, McGill, November 4th 2025

Summary

1. Benjamini-Schramm and Unimodularity
2. How to prove soficity (and previous results)
3. Minor-excluded graphs and main result
4. Open problems

Benjamini-Schramm convergence

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It has been studied in other settings, such as simplicial complexes, permutations, and manifolds.

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where R is the largest integer such that the balls $B_R(G, \rho)$ and $B_R(G', \rho')$ are isomorphic (as rooted graphs).

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This metric makes $\mathcal{G}_*$ into a complete and separable space. A **random rooted graph** is siply a random variable (G, ρ) taking values in $\mathcal{G}_*$.

Benjamini-Schramm/local convergence

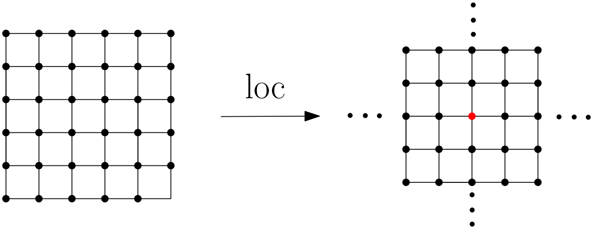
Definition. (Benjamini-Schramm/local convergence)

A sequence $G_1, G_2, \dots$ of finite graphs is said to converge **locally** to the random rooted graph (G, ρ) if: For all $r \in \mathbb{N}$ and every rooted graph H of radius r, we have that

$$\lim_{n\to\infty} \mathbb{P}_v \left[B_r(G_n, v) \cong H \right] = \mathbb{P} \left[B_r(G, \rho) \cong H \right] ,$$

where v denotes a uniform random vertex of G_n.

Examples of convergent sequences



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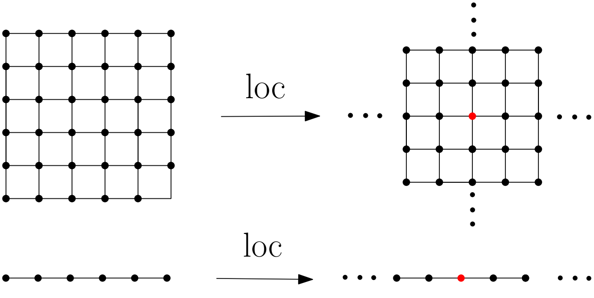

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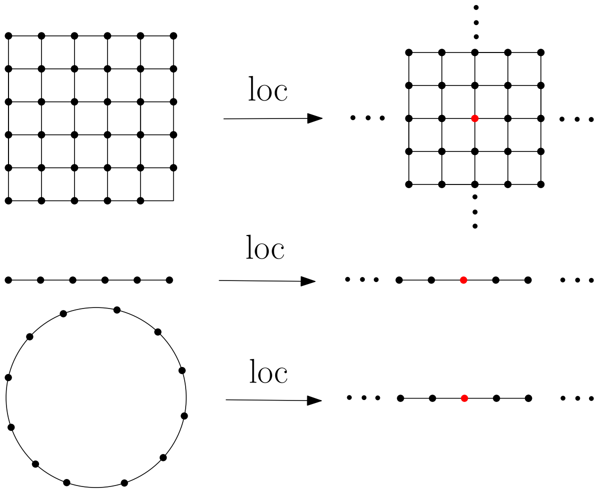

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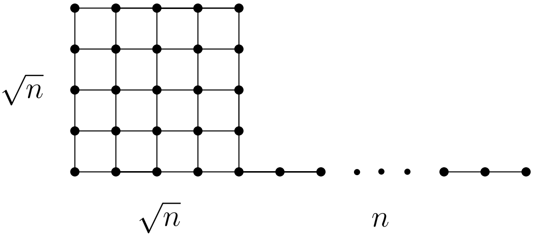

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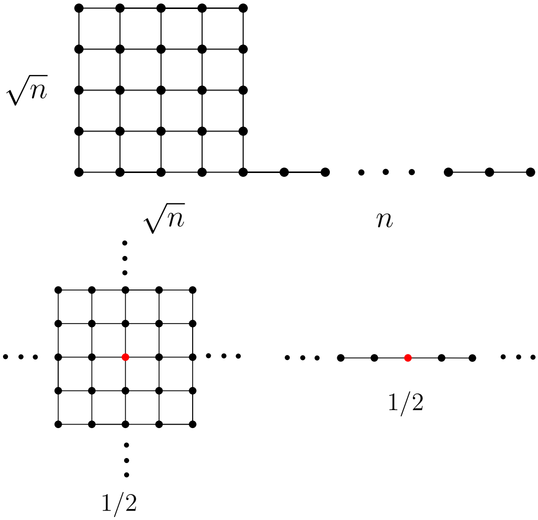

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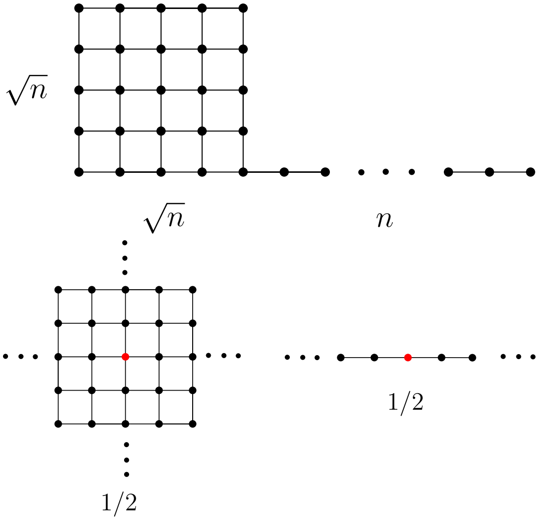

The distribution of the limiting random rooted graph has two atoms.

Soficity

Aldous-Lyons problem

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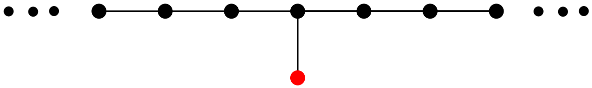
Not every random rooted graph is sofic:



Unimodularity (intuition)

Intuition

For a finite graph, each vertex has the same probability of being the root. In some sense, this property must be preserved under taking local limits.

Unimodularity (formal definition)

Definition. (Mass transport)

A *mass transport* is a Borel function f which takes as input a graph and two of its vertices, (G, ρ, o), and outputs a non-negative real number $f(G, \rho, o)$, indicating the amount of mass transported from ρ to o.

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Mass transport principle

A random rooted graph (G, ρ) is **unimodular** if it satisfies the *mass transport principle*: For every mass transport f, we have that

$$\mathbb{E}_{(G,\rho)}\left[\sum_{o\in V(G)} f(G,\rho,o)\right] = \mathbb{E}_{(G,\rho)}\left[\sum_{o\in V(G)} f(G,o,\rho)\right].$$

Aldous-Lyons problem

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It is not yet known whether there are non-sofic groups.
Conjectured non-sofic group: Higman group

$$a^{-1}ba = b^2, \quad b^{-1}cb = c^2, \quad c^{-1}dc = d^2, \quad d^{-1}ad = a^2\,.$$

Borel graphs and graphings

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Definition. $(\mathrm{Rel}(\mathcal{G}))$

A Borel graph $\mathcal{G}$ on X induces a **countable equivalence relation (CBER)** $\mathrm{Rel}(\mathcal{G}) \subseteq X^2$, which is defined by letting $x \sim y$ if and only if x and y belong to the same connected component of $\mathcal{G}$.

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Definition. (Graphings)

A **graphing** on (X, μ) is a Borel graph $\mathcal{G} \subseteq X^2$ such that, for any two Borel sets $A, B \subseteq X$, we have

$$\int_A \deg_B(x)\mu(dx) = \int_B \deg_A(x)\mu(dx) \,.$$

Graphings and unimodular graphs

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Theorem. (Unimodular graphs $\Leftrightarrow$ Graphings)

For every graphing $\mathcal{G}$ on (X, μ), the corresponding random rooted graph is unimodular. Furthermore, every unimodular random rooted graph arises from a graphing in this manner.

So we can talk about soficity for graphings.

Known tools for proving soficity

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- **Algebraic:** Wide range of tools and examples. Virtually free and virtually nilpotent groups, residually finite groups, soficity is preserved under both free and direct products, group theoretic bounded-degree HDX constructions, etc.
- **Non-Algebraic:** Almost all known proofs exploit one (or both) of the following properties
  - Hyperfiniteness/amenability (e.g., graphs of polynomial growth)
  - Treeability (e.g., trees and graphs with tree-like structure)

Hyperfinite graphings

Definition. (Hyperfiniteness)

We say that a graphing $\mathcal{G}$ on (X, μ) is **hyperfinite** if there exists a sequence $X_1 \subseteq X_2 \subseteq \cdots \subseteq X$ of Borel sets with $\bigcup_{i=1}^{\infty} X_i = X$ such that the restriction of $\mathcal{G}$ to each X_i has finite connected components.

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Hyperfiniteness is closely related to amenability. In fact, these two properties can be shown to be equivalent in some settings.

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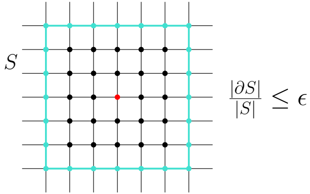
- For each ρ_i, look at its connected component G_i in $\mathcal{G}$. Then, cut off a sector S_i of vertices around the root with isoperimetric constant at most ϵ (i.e. small boundary). The neighborhoods of the elements of S_i that are far from the boundary remain unchanged.



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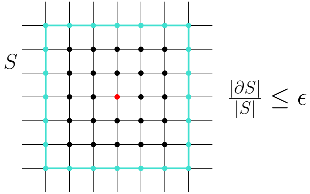

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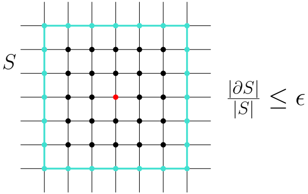

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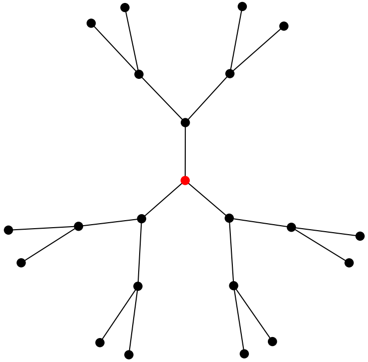
- Letting $\epsilon \to 0$ and $m \to \infty$, we get that $G_{\epsilon,m} \overset{\text{loc}}{\to} \mathcal{G}$.

Graphings with only acyclic components are sofic

Well known example: The d-regular tree is sofic, as it is the limit of random d-regular graphs (or random d-regular configuration models). A similar thing holds for every graphing all of whose components are trees: There is a sort of configuration model that converges to it.



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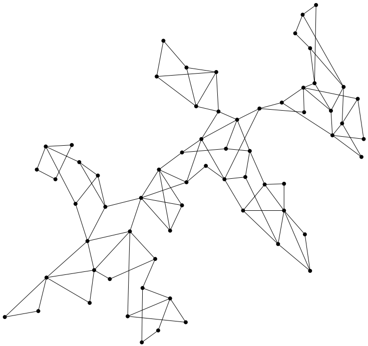
Theorem. (Elek, Lippner 2010)

Every treeable graphing is sofic.

Graphs which look like trees

Theorem. (Chen, Poulin, Tao, Tserunyan / Jardón-Sánchez 2023)

If $\mathcal{G}$ is a graphing all of whose connected components have **bounded treewidth**, then it is **treeable** ($\Rightarrow$sofic).



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In order to prove this result, Chen et al. study the space of *cuts* with *small boundary* $\mathcal{G}$. They exploit the fact that, for graphs of bounded treewidth, this space is very rich (*dense towards the ends*).

One-ended planar maps

Theorem. (Angel, Hutchcroft, Nachmias, Ray 2016 + Timár 2019)

If $\mathcal{G}$ is a graphing whose connected components are all **one-ended** and **planar**, and $\mathbb{E}_\mu \deg_{\mathcal{G}}(\rho) < \infty$, then $\mathcal{G}$ is treeable (and thus also sofic).

Note: A graph is **one-ended** if one **cannot** obtain two infinite connected components by deleting only finitely many vertices.

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This result relies on the connectivity of the *free uniform spanning forest* (FUSF). It is difficult to extend this technique to other setting (in particular, the proof relies heavily on planar duality).

Main results

Theorem. (SP 2025)

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The one-endedness is not required if all the connected components are also *quasi-transitive*. In fact, **H-minor-free+quasi-transitive** implies treeability.

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This explains why our result generalizes the earlier results for planar graphs.

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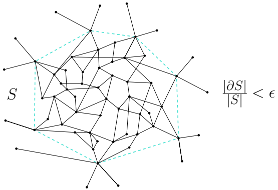

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This process can be repeated until the remaining graph has **expansion at least** ϵ, as desired.

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2. Delete a **small**, appropriately chosen (and possibly random) Borel set $\Psi = \Psi_{\epsilon,R} \subset X$ of vertices from $\mathcal{G}$, obtaining a new graphing $\mathcal{G}_{\text{thin}}$. Here, small means $\mu(\Psi) \leq \epsilon$.

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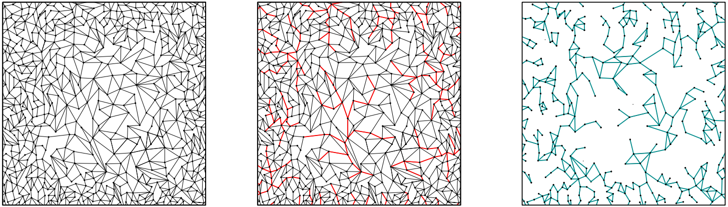

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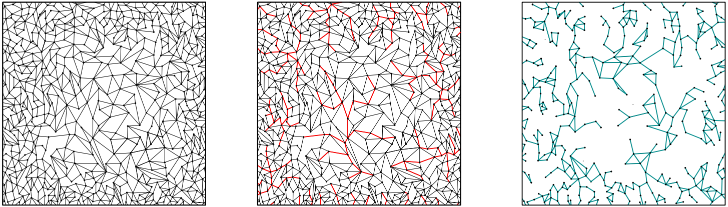

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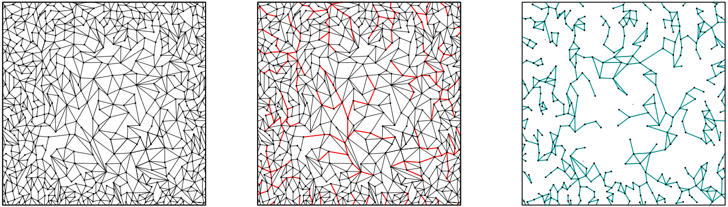

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These are the most important steps in the proof. The parts of the graph that are deleted in this step (shown in red) will be called **filaments**, due to their shape (more on this later).

Overview of the proof

Choose two parameters $\epsilon > 0$ and $R \in \mathbb{N}$.

1. Cut off large sections of the graph which have expansion factor less than ϵ. This allows us to reduce to the case where the graph has expansion at least ϵ.

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2. Delete a small, appropriately chosen (and possibly random) Borel set $\Psi = \Psi_{\epsilon,R} \subseteq X$ of vertices from $\mathcal{G}$, obtaining a new graphing $\mathcal{G}_{\text{thin}}$. Here, small means $\mu(\Psi) \leq \epsilon$.

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Local algorithms

In order to ensure the set is Borel, all modifications we make to the graph will be carried out by a **randomized local algorithm**, which decides whether a vertex should be deleted by observing only a ball of bounded radius around it (the algorithm also has access to some additional randomness).

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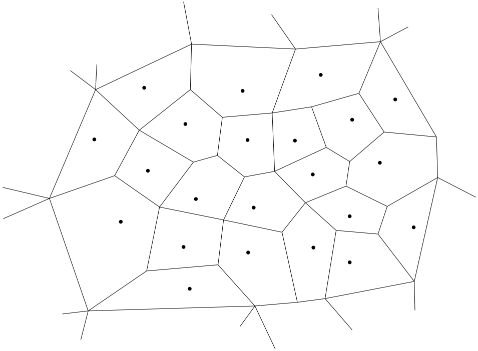
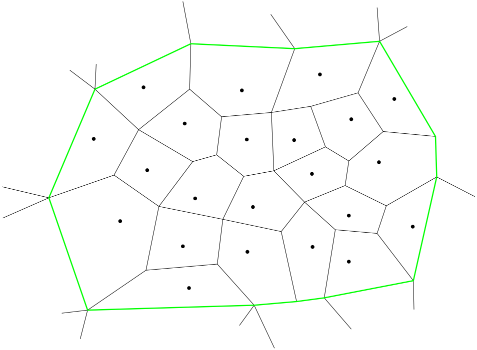
For every connected component G of $\mathcal{G}$, we do the following: sample a collection of $\Theta_\epsilon(1)$**-separated** vertices throughout the graph. Consider the **Voronoi decomposition** with respect to this collection of vertices.



Split the Voronoi cells into **large, finite, connected clusters**. The shape of these clusters does not matter too much.



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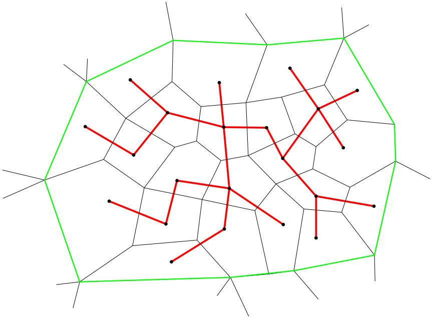
For each cluster, consider a minimal subtree of G which touches the centers of all the Voronoi cells it contains. These trees are called **filaments**.



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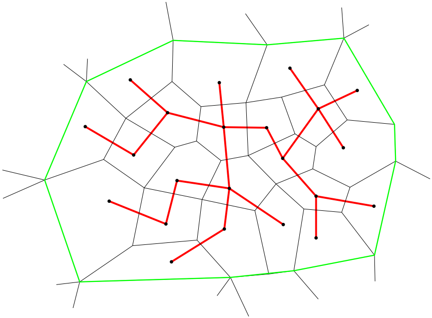

Let Ψ be the union of all the filaments over all sets and components of $\mathcal{G}$, and then delete Ψ from $\mathcal{G}$ to obtain the new graphing $\mathcal{G}_{\text{thin}}$.The ϵ-expansion of $\mathcal{G}$ can be used to show that, within each cluster (green), the fraction of vertices that belong to the filament (and thus to Ψ) is small. This yields $\mu(\Psi) \leq \epsilon$.

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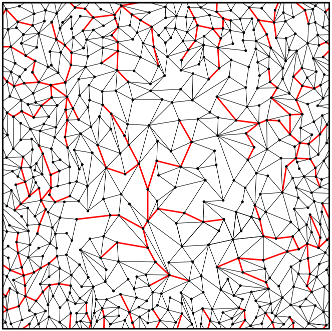

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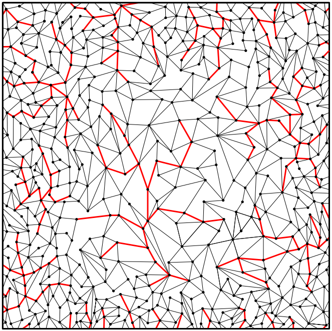

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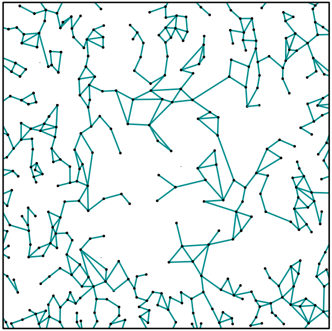

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We want to show that balls of radius R in G_{thin} have treewidth at most $O_\epsilon(1)$. For this, we will use the landmark grid-minor theorem.

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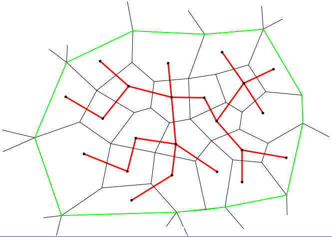
At a high level, this holds because the parts of the graph that lie in between the branches of each filament tree are *thin*. Thus, in order to find a $k \times k$ grid, we must use some long cycles which go *around the filament*.



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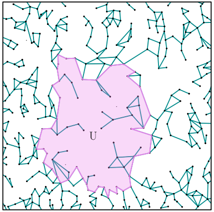
- Suppose B is a ball of radius R in G_{thin} which encloses a region U of G that, in turn, contains an entire filament F. Applying the ϵ-expansion property of G to U, B must have diameter at least

$$\epsilon|U| \geq \epsilon|F|,$$

so it suffices to make the filaments large enough that $\epsilon|F| \gg R$. This gives a contradiction, concluding the proof.



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- More generally: Find new ways to prove soficity (beyond combining amenable/hyperfinite + treeable).
- (Informal) In some sense, minor-excluded graphs have two-dimensional structure. Are all "finite-dimensional" one-ended graphs sofic? (Precise question question in this direction): If $\mathcal{G}$ is a graphing and each connected component the skeleton of some simplicial complex that is homeomorphic to $\mathbb{R}^3$, must $\mathcal{G}$ be sofic?
- Find some "non-algebraic", infinite, sofic, high-dimensional expander (HDX).

Galactic filaments